Alternative Discrete Energy Solutions to the Free Particle Dirac Equation

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Abstract

The usual method of solving the free particle Dirac equation results in the so called *continuum energy* solutions, with $|E| \geq mc^2$. Here, we take a different approach and find a set of solutions with quantized energies given by $E = \pm\, 2(j + 1/2)\, mc^2$.

Part 1: Solving the Dirac Equation as a Differential Equation

The free-particle Dirac equation

$$i\,\hbar\,\gamma^\mu \partial_\mu = m\,c\,\psi$$

Can be written in spherical coordinates as

$$\left(i\,\hbar\,\partial_0 - \gamma^0\,m\,c\right)\,\psi = \epsilon\,\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\frac{\sigma^i L_i}{r}\right)\,\psi \tag{1}$$

where

$$\sigma^i_{4\times 4} = \begin{bmatrix} \sigma^i_{2\times 2} & 0 \\ 0 & \sigma^i_{2\times 2} \end{bmatrix}$$

and $\sigma^i_{2\times 2}$ are the usual 2×2 Pauli matrices.

We demonstrate the transformation leading to (1) explicitly in Appendix A. By using the properties of the angular momentum operators we can write

$$\sigma^i L_i = \begin{bmatrix} L_3 & L_- & 0 & 0 \\ L_+ & -L_3 & 0 & 0 \\ 0 & 0 & L_3 & L_- \\ 0 & 0 & L_+ & -L_3 \end{bmatrix} \tag{2}$$

(When the 2×2 submatrices on the diagonal are identical, and the off-diagonal elements are null, we will sometimes alternate between the 2×2 and 4×4 version, depending on context). The 4×4 ϵ object is given by

$$\epsilon = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix}$$

with the 2×2 λ submatrices given by

$$\lambda = \left[\begin{array}{cc} cos\,\theta & sin\,\theta\, e^{-i\phi} \\ sin\,\theta\, e^{i\phi} & -\, cos\,\theta \end{array} \right]$$

We will now construct an *ansatz* to solve (1), seeking solutions which are eigentates of H, J^2, J_z.

First, consider the following *2-spinor* objects which can be constructed from spherical harmonics. The first is called θ^+:

$$\theta^+_{j,m_j} = \left[\begin{array}{c} \sqrt{j+m_j}\, Y_{j-1/2,m_j-1/2} \\ \sqrt{j-m_j}\, Y_{j-1/2,m_j+1/2} \end{array} \right] \tag{3}$$

And a second object that we call θ^-:

$$\theta^-_{j,m_j} = \left[\begin{array}{c} \sqrt{j+1-m_j}\, Y_{j+1/2,m_j-1/2} \\ -\sqrt{j+1+m_j}\, Y_{j+1/2,m_j+1/2} \end{array} \right] \tag{4}$$

We will see that these objects have total angular momentum $j = 1/2, 3/2....$ The index m_j is *half-integral* and runs from $m_j = j, ..., 1/2, -1/2, ... - j$. (Notice that the indices of the *spherical harmonics* are *integral* as required). (For a reference, see Bjorken and Drell, pg. 53).

These *2-spinors* obey the relations

$$\sigma^i L_i\, \theta^-_j = -\,\hbar(j+3/2)\theta^-_j$$

and

$$\sigma^i L_i\, \theta^+_j = \,\hbar(j-1/2)\theta^+_j$$

Notice how the $\theta^{+,-}$ behave slightly differently when acted upon by $\sigma^i L_i$, although they are each separately eigenvectors of it – this can be verified explicity by using the representation of $\sigma^i L_i$ shown in Eq. (2), which we demonstrate in Appendix B.

They both have the same total J^2:

$$J^2\, \theta^{+,-} = (L^2 + 3\hbar^2/4 + \hbar\, \sigma^i L_i)\, \theta^{+,-} = \hbar^2 j(j+1)\, \theta^{+,-}$$

They also both have the same value of $J_z = \hbar\, m_j$. Additionally they obey

$$\lambda\, \theta^-_j = \sqrt{\tfrac{j+1}{j}}\, \theta^+_j \quad \lambda\, \theta^+_j = \sqrt{\tfrac{j}{j+1}}\, \theta^-_j \tag{5}$$

That is, they are *conjugate* with respect to the λ operator. These last two identities follow from the spherical harmonic recursion relations and are demonstrated in Appendix B.

We find that we can construct two *4-spinor* solution classes to the Dirac Equation (1) from these:

$$\psi^e_j = e^{-iE_e t/\hbar} \left[\begin{array}{c} F^-_j(r)\, \theta^-_j \\ G^+_j(r)\, \theta^+_j \end{array} \right] \tag{6}$$

and

$$\psi^p_j = e^{-iE_p t/\hbar} \left[\begin{array}{c} F^+_j(r)\, \theta^+_j \\ G^-_j(r)\, \theta^-_j \end{array} \right] \tag{7}$$

where the $F^{+,-}_j(r)$ and $G^{+,-}_j(r)$ are *c-number* coefficients which are functions of radius only.

Let's investigate the first solution ψ_j^e and substitute (6) into the Dirac equation (1). We get:

$$\begin{bmatrix} (E_e/c - m\,c)F_j^-(r)\,\theta_j^- \\ (E_e/c + m\,c)G_j^+(r)\,\theta_j^+ \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ \lambda & 0 \end{bmatrix} \begin{bmatrix} (-i\,\hbar\partial/\partial r - i\,\hbar(j+3/2)/r)\; F_j^-(r)\,\theta_j^- \\ (-i\,\hbar\partial/\partial r + i\,\hbar(j-1/2)/r)\; G_j^+(r)\,\theta_j^+ \end{bmatrix}$$

which becomes

$$\begin{bmatrix} (E_e/c - m\,c)F_j^-(r)\,\theta_j^- \\ (E_e/c + m\,c)G_j^+(r)\,\theta_j^+ \end{bmatrix} = \begin{bmatrix} (-i\,\hbar\partial/\partial r + i\,\hbar(j-1/2)/r)\; G_j^+(r)\,\sqrt{\frac{j}{j+1}}\theta_j^- \\ (-i\,\hbar\partial/\partial r - i\,\hbar(j+3/2)/r)\; F_j^-(r)\,\sqrt{\frac{j+1}{j}}\,\theta_j^+ \end{bmatrix}$$

We can rearrange the terms and separate the matrix equation into two relations between the coefficients of the $\theta_j^{+,-}$:

$$\begin{aligned} (\partial/\partial r - (j-1/2)/r)\,G^+ &= \frac{(E_e/c - m\,c)\sqrt{\frac{j+1}{j}}}{-i\,\hbar}F^- \\ (\partial/\partial r + (j+3/2)/r)\,F^- &= \frac{(E_e/c + m\,c)\sqrt{\frac{j}{j+1}}}{-i\,\hbar}G^+ \end{aligned} \tag{8}$$

Now, the spherical Bessel (or Hankel) functions obey the coupled recursion relations:

$$\begin{aligned} (\partial/\partial r - (n-1)/r)\,j_{n-1} &= -\kappa\,j_n \\ (\partial/\partial r + (n+1)/r)\,j_n &= \kappa\,j_{n-1} \end{aligned} \tag{9}$$

If we make the index substitution $n = j + 1/2$, then these recursion relations become

$$\begin{aligned} (\partial/\partial r - (j-1/2)/r)\,j_{j-1/2} &= -\kappa\,j_{j+1/2} \\ (\partial/\partial r + (j+3/2)/r)\,j_{n+1/2} &= \kappa\,j_{j-1/2} \end{aligned} \tag{10}$$

So that we can solve the system given by Eq. (8) by identifying

$$F_j^- = j_{j+1/2}(\kappa r)\;,\; G_j^+ = e^{i\,\delta}\,j_{j-1/2}(\kappa r) \tag{11}$$

(allowing for an arbitrary phase between F and G). This means that the wavenumber κ must obey two relations:

$$-\,\kappa = e^{-i\delta}\,\frac{(E_e/c - m\,c)\,\sqrt{\frac{j+1}{j}}}{-i\,\hbar} \tag{12}$$

$$\kappa = e^{i\delta}\,\frac{(E_e/c + m\,c)\,\sqrt{\frac{j}{j+1}}}{-i\,\hbar} \tag{13}$$

which can be equated to each other to eliminate κ, giving the following relation satisfied by the energy E_e:

$$E_e = m\,c^2\frac{j+1+je^{i\,2\,\delta}}{j+1-je^{i\,2\,\delta}} \tag{14}$$

For the solution ψ_j^e to be a steady state solution, the energy E_e must be real. This restricts δ to either $\delta = \pi$ or $\delta = \pi/2$. In the first case $\delta = \pi$ we get:

$$E_e = 2\,(j+1/2)\,m\,c^2 \tag{15}$$

with corresponding κ from (12) or (13):

$$\kappa_j = -\frac{2m\,c}{\hbar}\sqrt{j(j+1)} \tag{16}$$

In the case $\delta = \pi/2$ we get solutions with an inverse relationship between energy and angular momentum, which for now we will dismiss as unphysical.

So by solving the Dirac equation in spherical coordinates, and requiring the solutions to be eigenstates of H, J^2 and J_z, we find that the energy is quantized. By using the solutions ψ_j^p, we find a similar quantization of energies, except now with negative energy values. We can summarize the solutions for both ψ_j^e and ψ_j^p in one expression:

$$E = \pm 2(j + 1/2)mc^2 \tag{17}$$

where the index $j = 1/2, 3/2, 5/2...$ with a degeneracy of $2\,j + 1$ for each index j.

Now we must address the discrepency between our solutions, and the traditional approach which results in the continuum of energy solutions. The beginning of the discrepancy comes in the choice of the definition of the $\theta^{+,-}$ *2-spinors*, as in Eqs. 3, 4. The traditional approach, as taken in *Bjorken & Drell*, [2], *Messiah*, [3], or just about any other textbook on advanced quantum mechanics is to define $\theta^{+,-}$ like this:

$$\theta^-_{j,m_j} = \frac{1}{\sqrt{2j+2}} \begin{bmatrix} \sqrt{j - m_j + 1}\, Y_{j+1/2,m_j-1/2} \\ -\sqrt{j + m_j + 1}\, Y_{j+1/2,m_j+1/2} \end{bmatrix}, \quad \theta^+_{j,m_j} = \frac{1}{\sqrt{2j}} \begin{bmatrix} \sqrt{j + m_j}\, Y_{j-1/2,m_j-1/2} \\ \sqrt{j - m_j}\, Y_{j-1/2,m_j+1/2} \end{bmatrix}$$

These additional prefactors of $\frac{1}{\sqrt{2j+2}}$ and $\frac{1}{\sqrt{2j}}$, result in *2-spinor* objects which are seperately *prenormalized* with respect to the angular variables:

$$\int \theta^\dagger \theta \, d\Omega = 1$$

In addition, instead of Eq. 5, the objects now obey a different *conjugate* relationship with respect to the λ operator:

$$\lambda\, \theta^+ = \theta^-, \quad \lambda\, \theta^- = \theta^+$$

This in effect *washes away* the the j dependant factors from the problem. Furthermore, instead of choosing a unitary phase factor $e^{i\delta}$ between F and G, as in Eq. 11, the traditional approach is to allow for a factor of arbitrary magnitude, call it v. So instead of Eqs. 12 and 13, we have:

$$\kappa = \frac{1}{v} \frac{(E/c - m\, c)}{i\, \hbar} \tag{18}$$

$$\kappa = v\, \frac{(E/c + m\, c)}{-i\, \hbar} \tag{19}$$

The j dependance is gone, and the factor v, related to the relative normalization between the upper two and lower two components introduces a third variable into a set of two equations with three unkowns. The procedure continues by choosing κ to be an arbitrary parameter which can take on any continuous value from 0 to $\pm\infty$, and then solving for E and v in terms of κ. E is determined from:

$$E(\kappa) = \pm\sqrt{(\hbar\kappa)^2 c^2 + m^2 c^4}$$

A continuous function of κ which has a minimum magnitude of mc^2. (This relation between E and κ is also satisfied by the discrete solutions, except that κ is fixed at discrete values).

So then, which approach is correct? One might proclaim that the traditional approach is to be favored because it is more in line with experimental fact. We know that free electrons do have continuous energies, as even the simplest cathode ray experiments seem to demonstrate. Furthermore, the existence of a spectrum of *heavy electrons* with masses of $2m$, $4m$, etc. has presumably not been observed. But as we will see later, these heavy electron states will still have the same *charge-to-mass* ratio of e/m, so they would

be indistinguishable from regular electrons in experiments using electric and magnetic fields. Furthermore, note that this solution approach which results in discrete energies has been undertaken in the *rest* frame of the electron. The inclusion of *Lorentz boosts* will still allow for a continuum of possible momenta and total energy.

To resolve the dillema, we must determine if the j-dependant factors which result from the conjugate relations between the $\theta^{+,-}$ as in Eq. 5 are inherently necessary and natural, or if a normalization which eliminates them is correct. To end the ambiguity, we will turn to a method of solving the Dirac equation other than solving it as a differential equation.

Part 2: Solving the Free-Particle Dirac Equation using Heisenberg Operator Methods

To begin, we note that the Dirac γ matrices can be viewed as representations of the four *unit vectors* of space-time. That is to say:

$$\hat{x}_0 = \gamma_0,\ \hat{x}_1 = \gamma_1,\ \hat{x}_2 = \gamma_2,\ \hat{x}_3 = \gamma_3 \tag{20}$$

These unit vectors obey:

$$\hat{x}_\mu \cdot \hat{x}_\nu = \frac{1}{2}\left(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu\right) = g_{\mu\nu} \tag{21}$$

Thus a *four-vector* object itself amounts to a 4×4 matrix:

$$\vec{A} = A^\mu\, \hat{x}_\mu = A^\mu\, \gamma_\mu \tag{22}$$

With this identification, the dot product and cross products between two vectors are represented by:

$$\vec{A} \cdot \vec{B} = \frac{1}{2}\{\vec{A}, \vec{B}\},\ \vec{A} \times \vec{B} = \frac{i}{2}\,\gamma^0\gamma^5\, \left[\vec{A}, \vec{B}\right] \tag{23}$$

The symmetric dot product of two vectors is proportional to their anticommutator, and the antisymmtric cross product of two vectors is related to their commutator. (An advantage of this representation of the unit vectors as matrices instead of as column vectors is that it allows for an algebraic formulation of the cross product which can be written on a two-dimensional piece of paper[1]).

We see that the Dirac operator $i\hbar\gamma^\mu\partial_\mu$ is the equivalent of the vector gradient operator $i\hbar\vec{\nabla}$. Thus the Dirac equation is the statement that the gradient of ψ is proportional to ψ:

$$i\hbar\vec{\nabla}\,\psi = mc\,\psi$$

It's interesting to note that in this matrix representation of vectors, the existence of spinor columns and rows which the vector operators can act upon is implicit. In other words, quantum mechanics is an implication of the matrix algebra of vectors.

Now, consider the four-vector which represents position:

$$\vec{x} = x^\mu\gamma_\mu$$

[1]For example, no algebraic operations beginning with $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ can result in $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The *four-velocity* can be calculated from the first derivative of this with respect to proper time:

$$\dot{\vec{x}} = \frac{d}{d\tau}\vec{x} = \dot{\vec{r}} + c\gamma^0$$

(In the rest frame of the electron, $\frac{d}{d\tau} = \frac{\partial}{\partial t}$). Now, note that $\vec{r} = r\hat{r} = -r\gamma^0\epsilon$. So again, taking the derivative with respect to time:

$$\dot{\vec{r}} = -\dot{r}\gamma^0\epsilon - r\gamma^0\dot{\epsilon} \tag{24}$$

We must now use the Heisenberg equation of motion to calculate $\dot{r}$ and $\dot{\epsilon}$:

$$\dot{r} = \frac{1}{i\hbar}\left[H, r\right], \dot{\epsilon} = \frac{1}{i\hbar}\left[H, \epsilon\right] \tag{25}$$

From Eq. (1) the Hamiltonian can be written:

$$H = \gamma^0\, mc^2 + c\,\epsilon\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\frac{\sigma^i L_i}{r}\right)$$

Remembering that r is a diagonal scalar, we can calculate $\dot{r}$ as

$$\begin{array}{rcl} \dot{r} &=& \frac{1}{i\hbar}\left[H, r\right] \quad = \quad \frac{1}{i\hbar}\left[\gamma^0\, mc^2 + c\,\epsilon\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\frac{\sigma^i L_i}{r}\right), r\right] \\ &=& -c\epsilon\left[\frac{\partial}{\partial r}, r\right] \\ &=& -c\epsilon \end{array} \tag{26}$$

Now, to calculate $\dot{\epsilon}$, we keep in mind that ϵ is a full matrix operator, but a function of the angular variables only:

$$\begin{array}{rcl} \dot{\epsilon} &=& \frac{1}{i\hbar}\left[H, \epsilon\right] \qquad = \qquad \frac{1}{i\hbar}\left[\gamma^0\, mc^2 + c\,\epsilon\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\frac{\sigma^i L_i}{r}\right), \epsilon\right] \\ &=& \cancelto{0}{\frac{1}{i\hbar}\left[-c\epsilon i\hbar\frac{\partial}{\partial r}, \epsilon\right]} + \frac{1}{i\hbar}\left[i\epsilon c\frac{\sigma^i L_i}{r}, \epsilon\right] + \frac{mc^2}{i\hbar}\left[\gamma^0, \epsilon\right] \\ &=& \frac{c}{\hbar}\left[\epsilon\frac{\sigma^i L_i}{r}, \epsilon\right] + \frac{mc^2}{i\hbar}\left[\gamma^0, \epsilon\right] \end{array} \tag{27}$$

We can now substitute the results of (26) and (27) into (24). Using $\gamma^0\epsilon = -\epsilon\gamma^0$, $\epsilon^2 = 1$ and $\gamma^0\epsilon - \epsilon\gamma^0 = 2\gamma^0\epsilon$, we obtain after some manipulation:

$$\dot{\vec{r}} = -c\gamma^0 + \frac{c}{\hbar}\gamma^0\left[\epsilon, \epsilon\,\sigma^i L_i\right] - \frac{2mc^2}{i\hbar}r\epsilon$$

Remembering $\dot{\vec{x}} = \dot{\vec{r}} + c\gamma^0$ we get an expression for the *four-velocity* vector operator.:

$$\dot{\vec{x}} = \frac{c}{\hbar}\gamma^0\left[\epsilon, \epsilon\,\sigma^i L_i\right] - \frac{2mc^2}{i\hbar}r\epsilon \tag{28}$$

We will be more interested in the *time-component* of this four vector, which we can obtain from:

$$\dot{x}_0 = \frac{1}{2}\{\dot{\vec{x}}, \gamma^0\}$$

After making use of facts like $\{\epsilon, \gamma^0\} = 0$ and $[\gamma^0, \sigma^i L_i] = 0$, we obtain after some manipulation:

$$\dot{x}_0 = \frac{c}{\hbar}\left(\sigma^i L_i - \epsilon\,\sigma^i L_i\,\epsilon\right) \tag{29}$$

Now let's investigate the action of the $\dot{x}_0$ operator on the spinor solutions of the Dirac equation. First, consider ψ_j^e, (suppressing time dependence of ψ):

$$\dot{x}_0\,\psi_j^e = \frac{c}{\hbar}\left(\sigma^i L_i - \epsilon\,\sigma^i L_i\,\epsilon\right)\left[\begin{array}{c} F^-\theta^- \\ G^+\theta^+ \end{array}\right] \tag{30}$$

Consider the operator $\epsilon = \left[\begin{array}{cc} 0 & \lambda \\ \lambda & 0 \end{array}\right]$. It is the action of λ on the $\theta^{-,+}$ which is the source of the ambiguity in the solutions of Part 1. That is:

$$\lambda\,\theta^- = s\,\theta^+, \lambda\,\theta^+ = \frac{1}{s}\,\theta^- \tag{31}$$

In the traditional approach, the factor $s = 1$. In Part 1, we chose to let $s = \sqrt{\frac{j+1}{j}}$. For now, let us leave s unspecified, and carry thru the calculation in (30):

$$\begin{aligned}
\dot{x}_0\,\psi_j^e &= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} -\hbar(j+3/2)F^-\theta^- \\ \hbar(j-1/2)G^+\theta^+ \end{array}\right] - \epsilon\,\sigma^i L_i \left[\begin{array}{c} 1/s\,G^+\theta^- \\ s\,F^-\theta^+ \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} -\hbar(j+3/2)F^-\theta^- \\ \hbar(j-1/2)G^+\theta^+ \end{array}\right] - \epsilon\left[\begin{array}{c} -(1/s)\,\hbar(j+3/2)G^+\theta^- \\ s\,\hbar(j-1/2)F^-\theta^+ \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} -\hbar(j+3/2)F^-\theta^- \\ \hbar(j-1/2)G^+\theta^+ \end{array}\right] - \left[\begin{array}{c} \hbar(j-1/2)F^-\theta^- \\ -\hbar(j+3/2)G^+\theta^+ \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left[\begin{array}{cc} -2\hbar(j+1/2) & 0 \\ 0 & 2\hbar(j+1/2) \end{array}\right]\left[\begin{array}{c} F^-\,\theta^- \\ G^+\,\theta^+ \end{array}\right] \\
&= -2(j+1/2)c\,\gamma_0\,\psi_j^e
\end{aligned} \tag{32}$$

Notice how the result is independent of the factor s. The double application of the operator ϵ in (30) results in factors of $\frac{1}{s}\times s$ causing the s dependence to disappear. However, since the j factors appear in a manner independant of the value of s, we can say that the presence of the j factors is natural and inherent.

Now let's calculate the action of $\dot{x}_0$ on ψ_j^p:

$$\begin{aligned}
\dot{x}_0\,\psi_j^p &= \tfrac{c}{\hbar}\left(\sigma^i L_i - \epsilon\,\sigma^i L_i\,\epsilon\right)\left[\begin{array}{c} F^+\theta^+ \\ G^-\theta^- \end{array}\right] \\
&= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} \hbar(j-1/2)F^+\theta^+ \\ -\hbar(j+3/2)G^-\theta^- \end{array}\right] - \epsilon\,\sigma^i L_i \left[\begin{array}{c} s\,G^-\theta^+ \\ (1/s)\,F^+\theta^- \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} \hbar(j-1/2)F^+\theta^+ \\ -\hbar(j+3/2)G^-\theta^- \end{array}\right] - \epsilon\left[\begin{array}{c} s\,\hbar(j-1/2)G^-\theta^+ \\ -(1/s)\,\hbar(j+3/2)F^+\theta^- \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left(\left[\begin{array}{c} \hbar(j-1/2)F^+\theta^+ \\ -\hbar(j+3/2)G^-\theta^- \end{array}\right] - \left[\begin{array}{c} -\hbar(j+3/2)F^+\theta^+ \\ \hbar(j-1/2)G^-\theta^- \end{array}\right]\right) \\
&= \tfrac{c}{\hbar}\left[\begin{array}{cc} 2\hbar(j+1/2) & 0 \\ 0 & -2\hbar(j+1/2) \end{array}\right]\left[\begin{array}{c} F^+\,\theta^+ \\ G^-\,\theta^- \end{array}\right] \\
&= 2(j+1/2)c\,\gamma_0\,\psi_j^p
\end{aligned} \tag{33}$$

Again, the s factors cancel out, and now we have a result which is proportional to $j+1/2$. In general we have:

$$\dot{x}_0\,\psi_j^{e,p} = \mp\,2(j+1/2)c\,\gamma_0\,\psi_j^{e,p} \tag{34}$$

That is, for the *electron* states ψ^e_j the negative sign pertains, while for the *positron* states, the sign is positive. We would almost be ready to interpret $\dot{x}_0$ as the operator E/mc if it were not for the fact that the overall sign is the opposite of that in Eq. (17). We will see in a moment that this should be so, and is due to the fact that electrons have *positive* energy, but *negative* charge, and positrons have *negative* energy but *positive* charge. Apparently, the correct identification of the E operator, to account for the minus sign is

$$E = -mc\,\gamma^0 \dot{x}_0 = \frac{mc^2}{\hbar}\,\gamma^0\,\left(\epsilon\,\sigma^i L_i\,\epsilon - \sigma^i L_i\right)$$

This operator is the Heisenberg counterpart to the operator $H = i\hbar\partial/\partial t$.

Let's see how this plays out further. Consider that the classical current is defined as

$$j_\mu = \rho\,\dot{x}_\mu$$

We can then see that there is a correspondence with this classical definition if we premultiply Eq. (34) with the electron charge e and $\bar{\psi}$:

$$\rho\,\dot{x}_0 = e\,\bar{\psi}^{e,p}\,\dot{x}_0 \psi^{e,p} = \mp e\,2(j+1/2)\,\bar{\psi}^{e,p}\,\gamma_0\,\psi^{e,p}$$

If we have already found a suitable normalization of ψ, (which will discuss in Part 3), then the term on the far right, which we know to be the normalized Dirac current $j_0 = \bar{\psi}\,\gamma_0\,\psi$, underscores this correspondence, and allows us to discern that the charges of these states are quantized as $q = \mp e\,2(j+1/2)$.

Now, let's again premultiply Eq. (34), but this time with $\psi^\dagger$ instead of $\bar{\psi}$:

$$\psi^{e,p\dagger}\,\dot{x}_0\,\psi^{e,p}_j = \mp\,2(j+1/2)c\,\psi^{e,p\dagger}\,\gamma_0\,\psi^{e,p}_j \tag{35}$$

The term on the far right is the invariant $\bar{\psi}\psi$, so therefore, the term on the left, which is the expectation value of $\dot{x}_0$ in the rest frame of the electron, must also be an invariant.

$$< \dot{x}_0 > = \int \psi^{e,p\dagger}\,\dot{x}_0\,\psi^{e,p} = \mp\,2(j+1/2)c\int \psi^{e,p\dagger}\,\gamma_0\,\psi^{e,p} \tag{36}$$

(For now, we have left the metric and bounds of the integration unspecified – we will deal with this in Part 3). If we can find a way to normalize the solutions $\psi^{e,p}$ such that:

$$\int \psi^{e,p\dagger}\gamma_0\psi^{e,p} = 1$$

Then we will have shown that

$$< \dot{x}_0 >_{e,p} = \mp\,2(j+1/2)c$$

Apparently, the correct correspondence principle between the energy and the expectation value of the zeroth component of *four-velocity* is

$$E_{e,p} = -mc\,< \dot{x}_0 >_{e,p}$$

(The presence of the minus sign would not be noticed in classical calculations, since they always involve the square of the energy or zeroth component of four-velocity). Thus the j dependent terms which we discovered in Part 1 are natural, and prenormalizing the $\theta^{+,-}$ spinors to eliminate j is inconsistent with this Heisenberg operator approach.

Thus the relation

$$E_{e,p} = \pm 2(j+1/2)m_0c^2$$

stands, and the ambiguity mentioned in Part 1 is resolved. We can interpret this as the prediction of a spectrum of heavy electrons and positrons, with inertial masses of $m = 2(j + 1/2)\, m_0$.

Note that since the charge of the state, q, is quantized in the same way as the rest-mass, the charge-to-mass ratio is constant:

$$\frac{q}{m} = \frac{\mp 2(j + 1/2)\, e}{2(j + 1/2)\, m_0} = \mp e/m_0$$

Which is consistent with the experimental fact that all electrons or positrons have the same charge-to-mass ratio.

Solution Table

<table>
<tr><td colspan="3">$E_e = 2(j + 1/2)mc^2$, $E_p = -2(j + 1/2)mc^2$, $\kappa_j = -\frac{2mc}{\hbar}\sqrt{j(j + 1)}$</td></tr>
<tr><td>$\psi_j^e = \begin{bmatrix} j_{j+1/2}(\kappa_j r)\theta_j^- \\ i j_{j-1/2}(\kappa_j r)\theta_j^+ \end{bmatrix} e^{-i\frac{E_e t}{\hbar}}$</td><td rowspan="2">$\theta_j^- = \begin{bmatrix} \sqrt{j - m_j + 1}\, Y_{j+1/2, m_j-1/2} \\ -\sqrt{j + m_j + 1}\, Y_{j+1/2, m_j+1/2} \end{bmatrix}$</td><td rowspan="2">$\theta_j^+ = \begin{bmatrix} \sqrt{j + m_j}\, Y_{j-1/2, m_j-1/2} \\ \sqrt{j - m_j}\, Y_{j-1/2, m_j+1/2} \end{bmatrix}$</td></tr>
<tr><td>$\psi_j^p = \begin{bmatrix} j_{j-1/2}(\kappa_j r)\theta_j^+ \\ i j_{j+1/2}(\kappa_j r)\theta_j^- \end{bmatrix} e^{-i\frac{E_p t}{\hbar}}$</td></tr>
</table>

Table 1: A table summarizing all the solutions of the free particle Dirac equation. For each j value, there is a degeneracy of 2j+1 .

Part 3: The Normalization of the Solutions over a Discrete Lattice

Now the volume integration of the spherical Bessel functions is divergent:

$$\int j_l^2(\kappa r) r^2 dr = \infty$$

So that the solutions to the *free particle* Dirac equation are not normalizable in the usual way. Both the traditional continuum solutions, and the solutions we presented in Part 1 have this feature. We are aware of the usual techniques used to deal with this divergence, such as limiting the extent of the volume integration, and then taking limits of ratios, etc., but our goal here is to introduce a new method of normalizing the free particle solutions.

All of the spherical Bessel functions have well defined, countable sets of zeroes. For example, $j_0(\kappa r) = \frac{\sin \kappa r}{\kappa r}$, is zero whenever $r = \frac{n\pi}{\kappa}$, $n = 1, 2, 3....$ Higher order Bessel functions like $j_1(\kappa r) = \frac{\sin \kappa r}{(\kappa r)^2} - \frac{\cos \kappa r}{\kappa r}$, have sets of zeroes for which there aren't simple formulae like for j_0, but they are nonetheless also discrete and countably infinite.

Interestingly, although no simple formula for the zeros of j_1 exists, there is a simple result when j_1 is evaluated *at the zeroes of* j_0. That is:

$$j_1(\kappa \frac{n\pi}{\kappa}) = j_1(n\pi) = \frac{\sin n\pi}{(n\pi)^2} - \frac{\cos n\pi}{n\pi} = (-1)^{n+1}\frac{1}{n\pi}$$

This allows us to conclude:

$$\sum_{n=1}^{\infty} j_1^2(\kappa \frac{n\pi}{\kappa}) = \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{\pi^2}\cdot\frac{\pi^2}{6} = \frac{1}{6}$$

In addition it is also true that

$$\sum_{n=1}^{\infty} j_0^2(\kappa \frac{n\pi}{\kappa}) = 0$$

since each term in the sum is zero.

Now consider the $j = 1/2$, positive energy, negative charge, electron solution:

$$\psi^e_{j=1/2} = \begin{bmatrix} j_1(\kappa r)\, \theta^-_{1/2} \\ i\, j_0(\kappa r)\, \theta^+_{1/2} \end{bmatrix}$$

Noting that $\int_{4\pi} \theta_j^{-\dagger} \theta_j^- \, d\Omega = 2j + 2$ and that $\int_{4\pi} \theta_j^{+\dagger} \theta_j^+ \, d\Omega = 2j$, we see that

$$\sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} {\psi^e_{1/2}}^\dagger \psi^e_{1/2} \, d\Omega = \frac{1}{6} \cdot (1+2) + 0 \cdot (1) = \frac{1}{2}$$

Similarly, for the the $j = 1/2$, negative energy, positive charge, positron solution:

$$\psi^p_{j=1/2} = \begin{bmatrix} j_0(\kappa r)\, \theta^+_{1/2} \\ i\, j_1(\kappa r)\, \theta^-_{1/2} \end{bmatrix}$$

We get that:

$$\sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} {\psi^p_{1/2}}^\dagger \psi^p_{1/2} \, d\Omega = 0 \cdot (1) + \frac{1}{6} \cdot (1+2) = \frac{1}{2}$$

So, both results give $1/2$:

$$\sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} {\psi^{e,p}_{1/2}}^\dagger \psi^{e,p}_{1/2} \, d\Omega = \frac{1}{2}$$

We can also calculate the value of $\bar{\psi} \cdot \psi$ when summed over the lattice of $n\pi$. For the electron state $\psi^e_{1/2}$, we get

$$\sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} \bar{\psi}^e_{1/2} \psi^e_{1/2} \, d\Omega = \sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} {\psi^e_{1/2}}^\dagger \gamma^0 \psi^e_{1/2} \, d\Omega = \frac{1}{6} \cdot (1+2) - 0 \cdot (1) = \frac{1}{2}$$

But for the positron state $\psi^p_{1/2}$ we get:

$$\sum_{r=\frac{n\pi}{\kappa}} \int_{4\pi} \bar{\psi}^p_{1/2} \psi^p_{1/2} \, d\Omega = 0 \cdot (1) - \frac{1}{6} \cdot (1+2) = -\frac{1}{2}$$

The opposite sign, but same magnitude! Apparently, the invariant quantity $\bar{\psi} \cdot \psi$ is related to the (negative of the) invariant charge of the state.

Now, so far, the fact that these sums converge for $j = 1/2$ is interesting, but this approach is not really useful as a normalization scheme, unless we can prove that *all* solutions ψ_j can be normalized in this way. May we find a formula:

$$\sum_{\text{zeroes of } j_0} \int_{4\pi} \bar{\psi}^{e,p}_j \psi^{e,p}_j \, d\Omega = d_j \tag{37}$$

That is, can we prove that the sums converge to a finite coefficient for all the solutions of the free particle Dirac equation? Discovering a general formula for the d_j would be nice, but not necessary. We will see

that the recursion formula for the Bessel functions will allow us to discover any d_j by building up from $d_{1/2}$. Expanding the above formula a little more explicitly we have for electrons

$$d_j^e = (2j+2) \cdot \sum_{n=1}^{\infty} j_{j+1/2}^2(n\pi) - (2j) \cdot \sum_{n=1}^{\infty} j_{j-1/2}^2(n\pi) \tag{38}$$

and for positrons:

$$d_j^p = (2j) \cdot \sum_{n=1}^{\infty} j_{j-1/2}^2(n\pi) - (2j+2) \cdot \sum_{n=1}^{\infty} j_{j+1/2}^2(n\pi) \tag{39}$$

which makes it obvious that $d_j^p = -d_j^e$. We also see that to prove that the d_j converge we must investigate the properties of $\sum_{n=1}^{\infty} j_{j\pm1/2}^2(n\pi)$.

Consider that the spherical Bessel functions obey the following recursion formula:

$$j_n(x) = \frac{2n-1}{x} j_{n-1}(x) - j_{n-2}(x) \tag{40}$$

This allows us to relate the value of any j_n back to the values of j_1 and j_0, by iteratively applying the above recursion relation. If we choose to evaluate j_n at one of the zeroes of j_0, the result becomes even simpler. For example, for $j_2(n\pi)$ we have:

$$j_2(n\pi) = \frac{2 \cdot 2 - 1}{n\pi} j_1(n\pi) - \cancelto{0}{j_0(n\pi)} = \frac{3}{n\pi} \frac{(-1)^{n+1}}{n\pi} = \frac{(-1)^{n+1}3}{(n\pi)^2}$$

Cleary, j_2 can also be normalized over the integer lattice defined by the zeores of j_0:

$$\sum_{n=1}^{\infty} j_2^2(n\pi) = \frac{9}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{9}{\pi^4} \cdot \frac{\pi^4}{90} = \frac{1}{10}$$

Just glancing at Eq. 40 allows us to infer that since $\sum_{n=1}^{\infty} j_2^2(n\pi)$ and $\sum_{n=1}^{\infty} j_1^2(n\pi)$ both converge, any sum involving higher j_n must also converge, since the power of $n\pi$ in the denominator is always higher. But can we discover a general formula? Perhaps, and I have not yet done so, but I don't need to to make an intriguing observation. We have calculated $d_{1/2} = 1/2$. Let's continue on and calculate $d_{3/2}$:

$$d_{3/2}^e = (2 \cdot \frac{3}{2} + 2) \cdot \sum_{n=1}^{\infty} j_2^2(n\pi) - (2 \cdot \frac{3}{2}) \cdot \sum_{n=1}^{\infty} j_1^2(n\pi) = 5 \cdot \frac{1}{10} - 3 \cdot \frac{1}{6} = 0$$

An interesting coincidence this is, how the terms involving the Bessel sums interacted with the terms resulting from the integration of the angular variables to result in exactly zero. If we continue on in this manner, we find that $d_{5/2}$, $d_{7/2}$, $d_{9/2}$ are all also zero. The question that arises then is *are all the d_j other than $d_{1/2}$ zero?*. Since I don't have a general formula for the Bessel sums, I have to turn to numerical methods to answer this question. And the answer I have found is: almost all of the d_j are zero, except for a very sparse subset. The next non-zero d_j after $d_{1/2}$ occurs when $j = 498\frac{1}{2}$, followed immediately by $j = 499\frac{1}{2}$. All the other d_j in between, at least as far as the Mathematica routines tell me, are exactly zero! Apparently, these states lying in between $j = 1/2$ and $j = 498\frac{1}{2}$ have an invariant norm which, defined as in Eq. 37, is zero. An implication is that, since these states have zero norm, we might expect their existence to be suppressed. Continuing beyond $j = 499\frac{1}{2}$, the d_j are again entirely zero, until we get to $j = 574\frac{1}{2}$ and then $j = 575\frac{1}{2}$. See Table 2 for a list of the first few non-zero d_j.

j	d_j
$1/2$	$1/2$
$498\,1/2$	166.38
$499\,1/2$	-166.38
$574\,1/2$	1.05071×10^{136}
$575\,1/2$	-1.05071×10^{136}
$653\,1/2$	1.59431×10^{240}
$654\,1/2$	-1.59431×10^{240}

Table 2: A table of the first few non-zero d_j.

If there is any validity to this normalization scheme, then these results seem to imply that most of the particle solutions $E = \pm 2(j + 1/2)mc^2$ are suppressed, except for a very sparse spectrum with non-zero norms d_j. Now, we know from the results of particle physics that a sparse spectrum of particles does indeed exist, but I have not yet been able to match up the known particles such as the muons, pions, etc. with these d_j. However, I have still presented this method hoping that it, or a slight modification of it, may be able to predict the known experimental spectrum of particles. I know that this is not the complete answer yet, because it results only from the free particle Dirac equation in the absence of any potentials. In a future work I hope to consider the effect of different types of potentials A_μ on the solution spectrum and the resulting d_j.

Conclusion

Now, the result of the bound-state *Coulomb*-potential Dirac equation is that the internal energies of the Hydrogen atom are quantized as $E_n \approx mc^2 - \frac{13.6eV}{n^2}$. Thus, the energy of bound-state electrons are very nearly $E = mc^2$, for even the most deeply bound $n = 1$ electrons. I have found that the factors of $\sqrt{\frac{j+1}{j}}$ which resulted in the ambiquity mentioned in Part 1, do not come into play in the Hydrogen atom, and so my conclusion is that the traditional solutions to the Hydrogen atom problem are correct, as they must be (excluding hyperfine considerations). In a future work, I will apply the Heisenberg operator method to the problem of the Hydrogen atom, and show that it gives the same result as directly solving the differential equation, as I did in Part 2 of this work. We will see that since the covariant operator H undergoes a transformation $H \to i\hbar\frac{\partial}{\partial t} - \frac{e^2}{r}$ as a result of the Coulomb potential, there must be a corresponding transformation of the contravariant Heisenberg operator $\dot{x}^0$. Discovering the laws of transformation of the Heisenberg operators $\dot{\vec{x}}$ which result from the transformation of the covariant $P_\mu \to i\hbar\partial_\mu + \frac{q}{c}A_\mu$ will be one central goal of a subsequent article.

Finally, the results from Parts 1 and 2 show that the lowest energy of a free particle state is $E = 2(1/2 + 1/2)mc^2 = 2mc^2$, so that the first excited state free electron, has an energy of mc^2 greater than a typical bound electron: $2mc^2 - mc^2 = mc^2$. But experimental work with cathode rays, the photoelectric effect and other elementary experiments seem to imply that free electrons have an energy always greater than mc^2, say $E_{\text{free}} = mc^2 + \Delta E$, where ΔE can range from a few eV, to several hundred-thousands of eV, at which point the free electrons begin to emit γ-rays and cause pair production. We might instead consider these free electrons to have energies of $E = 2mc^2 - \Delta E'$. Where $\Delta E'$ results from the perturbation of the free-particle state by a potential field. Again, we will address this in more detail in a subsequent article.

Appendix A

To demonstrate the transformations leading to Eq. (1), we will use the properties of the γ and σ matrices and their identities. We will work in the representation of the γ used by Bjorken and Drell [2]:

$$\gamma^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \gamma^i = \begin{bmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{bmatrix} \quad \gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{41}$$

and use the typical representation of the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{42}$$

The Pauli matrices obey the identities

$$\begin{array}{ccc} \sigma^1\,\sigma^2 = i\,\sigma^3 & \sigma^3\,\sigma^1 = i\,\sigma^2 & \sigma^2\,\sigma^3 = i\,\sigma^1 \\ \sigma^2\,\sigma^1 = -i\,\sigma^3 & etc. & \end{array} \tag{43}$$

so therefore the γ obey the identities:

$$\begin{array}{ccc} \gamma^1\,\gamma^2 = -i\,\sigma^3 & \gamma^3\,\gamma^1 = -i\,\sigma^2 & \gamma^2\,\gamma^3 = -i\,\sigma^1 \\ \gamma^2\,\gamma^1 = i\,\sigma^3 & etc. & \end{array} \tag{44}$$

Here, the σ are 4×4 block-diagonal matrices formed from the 2×2 Pauli matrices. The γ and σ also have the properties:

$$(\sigma^i)^2 = 1 \quad (\gamma^0)^2 = 1 \quad (\gamma^i)^2 = -1 \tag{45}$$

Now consider the following:

$$\begin{aligned} (\gamma^j\,x^j)(\gamma^i\,\partial_i) \;&= (\gamma^1)^2\,x^1\partial_1 + \gamma^1\,\gamma^2\,x^1\partial_2 + ... \\ &\quad + (\gamma^2)^2\,x^2\partial_2 + \gamma^2\,\gamma^1\,x^2\partial_1 + ... \end{aligned}$$

which by virtue of the identities (44) and (45) becomes

$$\begin{aligned} (\gamma^j\,x^j)(\gamma^i\,\partial_i) \;&= -x^i\partial_i + i\,\sigma^1\,(x^3\partial_2 - x^2\partial_3) + ... \\ &= -r\,\partial/\partial r + \sigma^i L_i/\hbar \end{aligned}$$

so that we have established the identity

$$(\gamma^j\,x^j)(-i\,\hbar\,\gamma^i\,\partial_i) = i\,\hbar\,r\frac{\partial}{\partial r} - i\,\sigma^i L_i$$

which we will find convinient to recast as

$$\left(\frac{\gamma^0\,\gamma^j\,x^j}{r}\right)\left(-i\,\hbar\,\gamma^0\,\gamma^i\,\partial_i\right) = -i\,\hbar\,\frac{\partial}{\partial r} + i\,\frac{\sigma^i L_i}{r} \tag{46}$$

When we write the $\gamma^j\,x^j$ object explicitly as

$$\gamma^j\,x^j = \begin{bmatrix} 0 & 0 & x^3 & x^1 - i\,x^2 \\ 0 & 0 & x^1 + i\,x^2 & -x^3 \\ -x^3 & -x^1 + i\,x^2 & 0 & 0 \\ -x^1 - i\,x^2 & x^3 & 0 & 0 \end{bmatrix} \tag{47}$$

it becomes apparent that

$$\left(\gamma^j\, x^j\right)^2 = -r^2\, 1_{4\times 4}$$

so that we can make the following statement about the inverse matrix

$$\left(\frac{\gamma^j\, x^j}{r}\right)^{-1} = -\left(\frac{\gamma^j\, x^j}{r}\right) \tag{48}$$

Then, using (48) with (46) we obtain

$$-\, i\,\hbar\gamma^0\,\gamma^i\,\partial_i = \frac{\gamma^0\,\gamma^j\,x^j}{r}\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\,\frac{\sigma^i L_i}{r}\right) \tag{49}$$

so that the Dirac equation

$$\left(i\,\hbar\,\partial_0 - mc\,\gamma^0\right)\;\psi = -i\,\hbar\,\gamma^0\,\gamma^i\,\partial_i\,\psi$$

becomes

$$\left(i\,\hbar\,\partial_0 - mc\,\gamma^0\right)\;\psi = \epsilon\,\left(-i\,\hbar\,\frac{\partial}{\partial r} + i\,\frac{\sigma^i L_i}{r}\right)\;\psi$$

This establishes Eq. (1), with the object ϵ defined as

$$\epsilon \equiv \frac{\gamma^0\,\gamma^j\,x^j}{r} \tag{50}$$

When the explicit form of $\gamma^j\,x^j$ from (47) is combined with the definition of the spherical coordinates

$$\begin{aligned} x^1 \; &= r\,sin\,\theta\,cos\,\phi \\ x^2 \; &= r\,sin\,\theta\,sin\,\phi \\ x^3 \; &= r\,cos\,\theta \end{aligned}$$

We find that the explicit form of ϵ becomes

$$\epsilon = \begin{bmatrix} 0 & 0 & cos\,\theta & sin\,\theta\,e^{-i\,\phi} \\ 0 & 0 & sin\,\theta\,e^{i\,\phi} & -cos\,\theta \\ cos\,\theta & sin\,\theta\,e^{-i\,\phi} & 0 & 0 \\ sin\,\theta\,e^{i\,\phi} & -cos\,\theta & 0 & 0 \end{bmatrix} \tag{51}$$

Note how in the definition of the ϵ object, Eq. 50 and in other relations, the summation indicies j are both raised in the $\gamma^j x^j$ object. In a simple flat space metric, this does not create a problem, so long as we note that since $\vec{r} = \gamma^j x_j$, then $\epsilon = -\gamma_0 \hat{r}$.

Appendix B

Here we will establish the properties of the $\theta^{+,-}$ *2-spinors* when acted upon by λ and $\sigma^i L_i$. We will make use of the the properties of the angular momentum operators and spherical harmonics:

$$\begin{aligned} L_+\,Y_{l,m} \; &= \sqrt{(l-m)(l+m+1)}\,Y_{l,m+1} \\ L_-\,Y_{l,m} \; &= \sqrt{(l+m)(l-m+1)}\,Y_{l,m-1} \end{aligned} \tag{52}$$

$$
\begin{aligned}
\sigma^i L_i\, \theta_j^+ &= \begin{bmatrix} L_z & L_- \\ L_+ & -L_z \end{bmatrix} \begin{bmatrix} \sqrt{j+m_j}\, Y_{j-1/2,m_j-1/2} \\ \sqrt{j-m_j}\, Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \sqrt{j+m_j}(m_j-1/2)\, Y_{j-1/2,m_j-1/2} + \sqrt{j-m_j}\,\sqrt{(j-m_j)(j-1/2-m_j-1/2+1)}\, Y_{j-1/2,m_j-1/2} \\ \sqrt{j+m_j}\,\sqrt{(j-1/2-m_j+1/2)(j-1/2+m_j-1/2+1)}\, Y_{j-1/2,m_j+1/2} - (m_j+1/2)\,\sqrt{j-m_j}\, Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \left((m_j-1/2)\sqrt{j+m_j} + \sqrt{j-m_j}\sqrt{(j+m_j)(j-m_j)}\right) Y_{j-1/2,m_j-1/2} \\ \left(\sqrt{j+m_j}\sqrt{(j-m_j)(j+m_j)} - (m_j+1/2)\sqrt{j-m_j}\right) Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \sqrt{j+m_j}(m_j-1/2+j-m_j) Y_{j-1/2,m_j-1/2} \\ \sqrt{j-m_j}(j+m_j-m_j-1/2) Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar\,(j-1/2)\,\theta_j^+
\end{aligned}
\tag{53}
$$

$$
\begin{aligned}
\sigma^i L_i\, \theta_j^- &= \begin{bmatrix} L_z & L_- \\ L_+ & -L_z \end{bmatrix} \begin{bmatrix} \sqrt{j-m_j+1}\, Y_{j+1/2,m_j-1/2} \\ -\sqrt{j+m_j+1}\, Y_{j+1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \sqrt{j-m_j+1}(m_j-1/2)\, Y_{j+1/2,m_j-1/2} - \sqrt{j+m_j+1}\,\sqrt{(j+1/2+m_j+1/2)(j+1/2-m_j-1/2+1)}\, Y_{j+1/2,m_j-1/2} \\ \sqrt{j-m_j+1}\,\sqrt{(j+1/2-m_j+1/2)(j+1/2+m_j-1/2+1)}\, Y_{j+1/2,m_j+1/2} + (m_j+1/2)\,\sqrt{j+m_j+1}\, Y_{j+1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \left((m_j-1/2)\sqrt{j-m_j+1} - \sqrt{j+m_j+1}\sqrt{(j+m_j+1)(j-m_j+1)}\right) Y_{j+1/2,m_j-1/2} \\ \left(\sqrt{j-m_j+1}\sqrt{(j-m_j+1)(j+m_j+1)} + (m_j+1/2)\sqrt{j+m_j+1}\right) Y_{j+1/2,m_j+1/2} \end{bmatrix} \\
&= \hbar \begin{bmatrix} \sqrt{j-m_j+1}(m_j-1/2-j-m_j-1) Y_{j+1/2,m_j-1/2} \\ \sqrt{j+m_j+1}(j-m_j+1+m_j+1/2) Y_{j+1/2,m_j+1/2} \end{bmatrix} \\
&= -\hbar\,(j+3/2)\,\theta_j^-
\end{aligned}
\tag{54}
$$

Now use the following identities to calculate the action of λ:

$$
\begin{aligned}
\cos\theta\, Y_{l,m} &= \sqrt{\tfrac{(l-m+1)(l+m+1)}{(2l+1)(2l+3)}}\, Y_{l+1,m} + \sqrt{\tfrac{(l-m)(l+m)}{(2l-1)(2l+1)}}\, Y_{l-1,m} \\
\sin\theta\, e^{i\phi}\, Y_{l,m} &= -\sqrt{\tfrac{(l+m+1)(l+m+2)}{(2l+1)(2l+3)}}\, Y_{l+1,m+1} + \sqrt{\tfrac{(l-m)(l-m-1)}{(2l-1)(2l+1)}}\, Y_{l-1,m+1} \\
\sin\theta\, e^{-i\phi}\, Y_{l,m} &= \sqrt{\tfrac{(l-m+1)(l-m+2)}{(2l+1)(2l+3)}}\, Y_{l+1,m-1} - \sqrt{\tfrac{(l+m)(l+m-1)}{(2l-1)(2l+1)}}\, Y_{l-1,m-1}
\end{aligned}
\tag{55}
$$

$$
\begin{aligned}
\lambda\, \theta_j^- &= \begin{bmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{bmatrix} \begin{bmatrix} \sqrt{j-m_j+1}\, Y_{j+1/2,m_j-1/2} \\ -\sqrt{j+m_j+1}\, Y_{j+1/2,m_j+1/2} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{j-m_j+1}\left(\sqrt{\frac{(j+1/2-m_j+1/2+1)(j+1/2+m_j-1/2+1)}{(2j+2)(2j+4)}}\, Y_{j+3/2,m_j-1/2} + \sqrt{\frac{(j+1/2-m_j+1/2)(j+1/2+m_j-1/2)}{(2j)(2j+2)}}\, Y_{j-1/2,m_j-1/2}\right) \\ -\sqrt{j+m_j+1}\left(\sqrt{\frac{(j+1/2-m_j-1/2+1)(j+1/2-m_j-1/2+2)}{(2j+2)(2j+4)}}\, Y_{j+3/2,m_j-1/2} - \sqrt{\frac{(j+1/2+m_j+1/2)(j+1/2+m_j+1/2-1)}{(2j)(2j+2)}}\, Y_{j-1/2,m_j-1/2}\right) \\ \sqrt{j-m_j+1}\left(-\sqrt{\frac{(j+1/2+m_j-1/2+1)(j+1/2+m_j-1/2+2)}{(2j+2)(2j+4)}}\, Y_{j+3/2,m_j+1/2} + \sqrt{\frac{(j+1/2-m_j+1/2)(j+1/2-m_j+1/2-1)}{(2j)(2j+2)}}\, Y_{j-1/2,m_j+1/2}\right) \\ +\sqrt{j+m_j+1}\left(\sqrt{\frac{(j+1/2-m_j-1/2+1)(j+1/2+m_j+1/2+1)}{(2j+2)(2j+4)}}\, Y_{j+3/2,m_j+1/2} + \sqrt{\frac{(j+1/2-m_j-1/2)(j+1/2+m_j+1/2)}{(2j)(2j+2)}}\, Y_{j-1/2,m_j+1/2}\right) \end{bmatrix} \\
&= \begin{bmatrix} \left((j-m_j+1)\sqrt{\frac{j+1/2+m_j-1/2}{(2j+2)(2j)}} + (j+m_j+1)\sqrt{\frac{j+1/2+m_j-1/2}{(2j+2)(2j)}}\right) Y_{j-1/2,m_j-1/2} \\ \left((j-m_j+1)\sqrt{\frac{j+1/2-m_j-1/2}{(2j)(2j+2)}} + (j+m_j+1)\sqrt{\frac{j+1/2-m_j-1/2}{(2j+2)(2j)}}\right) Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{\frac{2j+2}{2j}}\,\sqrt{j+m_j}\, Y_{j-1/2,m_j-1/2} \\ \sqrt{\frac{2j+2}{2j}}\,\sqrt{j-m_j}\, Y_{j-1/2,m_j+1/2} \end{bmatrix} \\
&= \sqrt{\tfrac{j+1}{j}}\,\theta_j^+
\end{aligned}
\tag{56}
$$

Because $\lambda^2 = 1$ this then implies that

$$
\lambda\, \theta_j^+ = \sqrt{\frac{j}{j+1}}\,\theta_j^- \tag{57}
$$

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